COMMON FIXED POINT THEOREMS FOR PAIRS OF SUBCOMPATIBLE MAPS

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Abstract

In this paper, we introduce the new concepts of subcompatibility and subsequential continuity which are respectively weaker than occasionally weak compatibility and reciprocal continuity. With them, we establish a common fixed point theorem for four maps in a metric space which improves a recent result of Jungck and Rhoades [7]. Also we give another common fixed point theorem for two pairs of subcompatible maps of Greguš type which extends results of the same authors, Djoudi and Nisse [3], Pathak et al. [12] and others and we end our work by giving a third result which generalizes results of Mbarki [8] and others.

Key words and phrases: Commuting and weakly commuting maps, compatible and compatible maps of type (A), (B), (C) and (P), weakly compatible maps, occasionally weakly compatible maps, subcompatible maps, subsequentially continuous maps, coincidence point, common fixed point, Greguš type.

2000 Mathematics Subject Classification: 47H10, 54H25.

1 Historical introduction and new definitions

Let $(\mathcal{X}, d)$ be a metric space and let f and g be two maps from $(\mathcal{X}, d)$ into itself. f and g are commuting if $fgx = gfx$ for all x in $\mathcal{X}$.

To generalize the notion of commuting maps, Sessa [15] introduced the concept of weakly commuting maps. He defines f and g to be weakly commuting if

$$d(fgx, gfx) \leq d(gx, fx)$$

for all $x \in \mathcal{X}$. Obviously, commuting maps are weakly commuting but the converse is not true.

In 1986, Jungck [4] gave more generalized commuting and weakly commuting maps called compatible maps. f and g above are called compatible if

$$(1)\ \lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever (x_n) is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$. Clearly, weakly commuting maps are compatible, but the implication is not reversible (see [4]).

Afterwards, the same author with Murthy and Cho [6] made another generalization of weakly commuting maps by introducing the concept of compatible maps of type (A). Previous f and g are said to be compatible of type (A) if in place of (1) we have two following conditions

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(gfx_n, f^2x_n) = 0.$$

It is clear to see that weakly commuting maps are compatible of type (A), from [6] it follows that the implication is not reversible.

In their paper [11], Pathak and Khan extended type (A) maps by introducing the concept of compatible maps of type (B) and compared these maps with compatible and compatible maps of type (A) in normed spaces. To be compatible of type (B), f and g above have to satisfy, in lieu of condition (1), the inequalities

$$\begin{aligned}
\lim_{n\to\infty} d(fgx_n, g^2x_n) &\leq \frac{1}{2}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)\right] \\
&\quad \text{and} \\
\lim_{n\to\infty} d(gfx_n, f^2x_n) &\leq \frac{1}{2}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n)\right].
\end{aligned}$$

It is clear that compatible maps of type (A) are compatible of type (B), to show that the converse is not true (see [11]).

Further, in 1998, Pathak et al. [12] introduced another generalization of compatibility of type (A) by giving the concept of compatible maps of type (C). f and g are said to be compatible of type (C) if they satisfy the two inequalities

$$\begin{aligned}
\lim_{n\to\infty} d(fgx_n, g^2x_n) &\leq \frac{1}{3}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)\right. \\
&\quad \left. + \lim_{n\to\infty} d(ft, g^2x_n)\right] \\
&\quad \text{and} \\
\lim_{n\to\infty} d(gfx_n, f^2x_n) &\leq \frac{1}{3}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n)\right. \\
&\quad \left. + \lim_{n\to\infty} d(gt, f^2x_n)\right].
\end{aligned}$$

The same authors gave some examples to show that compatible maps of type (C) need not be neither compatible nor compatible of type (A) (resp. type (B)).

In [10] the concept of compatible maps of type (P) was introduced and compared with compatible and compatible maps of type (A). f and g are compatible of type (P) if instead of (1) we have

$$\lim_{n\to\infty} d(f^2x_n, g^2x_n) = 0.$$

Note that compatibility, compatibility of type (A) (resp. (B), (C) and (P)) are equivalent if f and g are continuous.

In his paper [5], Jungck generalized the compatibility, the compatibility of type (A) (resp. type (B), (C) and (P)) by introducing the concept of weak compatibility. He defines f and g to be weakly compatible if $ft = gt$ for some $t \in \mathcal{X}$ implies that $fgt = gft$.

It is known that all of the above compatibility notions imply weakly compatible notion, however, there exist weakly compatible maps which are neither compatible nor compatible of type (A), (B), (C) and (P) (see [1]).

Recently, Al-Thagafi and Shahzad [2] weakened the concept of weakly compatible maps by giving the new concept of occasionally weakly compatible maps (owc). Two self-maps f and g of a set $\mathcal{X}$ to be owc iff there is a point x in $\mathcal{X}$ which is a coincidence point of f and g at which f and g commute; i.e., there exists a point x in $\mathcal{X}$ such that $fx = gx$ and $fgx = gfx$.

In this paper, we weaken the above notion by introducing a new concept called **subcompatibility**.

1.1 Definition *Let $(\mathcal{X}, d)$ be a metric space. Maps f and $g : \mathcal{X} \to \mathcal{X}$ are said to be* **subcompatible** *iff there exists a sequence $(x_n)_n$ in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, $t \in \mathcal{X}$ and which satisfy $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$.*

Obviously, two owc maps are subcompatible, however the converse is not true in general. The example below shows that there exist subcompatible maps which are not owc.

1.2 Example Let $\mathcal{X} = [0, \infty[$ with the usual metric d. Define f and g as follows:

$$fx = x^2 \text{ and } gx = \begin{cases} x+2 & \text{if} \quad x \in [0,4] \cup]9, \infty[, \\ x+12 & \text{if} \quad x \in]4, 9]. \end{cases}$$

Let (x_n) be a sequence in $\mathcal{X}$ defined by $x_n = 2 + \dfrac{1}{n}$ for $n \in \mathbb{N}^* = \{1, 2, \ldots\}$. Then,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n^2 = 4 = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} (x_n + 2),$$

and

$$\begin{aligned} fgx_n &= f(x_n + 2) = (x_n + 2)^2 \to 16 \text{ when } n \to \infty \\ gfx_n &= g(x_n^2) = x_n^2 + 12 \to 16 \text{ when } n \to \infty \end{aligned}$$

thus, $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$; that is, f and g are subcompatible.

On the other hand, we have $fx = gx$ iff $x = 2$ and

$$\begin{aligned} fg(2) &= f(4) = 4^2 = 16 \\ gf(2) &= g(4) = 4 + 2 = 6 \end{aligned}$$

then, $f(2) = 4 = g(2)$ but $fg(2) = 16 \neq 6 = gf(2)$, hence f and g are not owc. ∎

Clearly, we can resume implications between previous notions by the following list:

- Commuting maps $\Rightarrow$ Weakly commuting maps
- Weakly commuting maps $\Rightarrow$ Compatible maps
- Weakly commuting maps $\Rightarrow$ Compatible maps of type (A)
- Compatible maps of type (A) $\Rightarrow$ Compatible maps of type (B)
- Compatible maps of type (A) $\Rightarrow$ Compatible maps of type (C)
- Compatible maps (resp. Compatible of type (A), (B), (C), (P)) $\Rightarrow$ Weakly compatible maps
- Weakly compatible maps $\Rightarrow$ Occasionally Weakly compatible maps
- Occasionally weakly compatible maps $\Rightarrow$ **Subcompatible maps**.

Now, our second objective is to introduce a new notion called **subsequential continuity** which weakens the concept of reciprocal continuity which was introduced by Pant in his paper [9], as follows: Self-maps f and g of a metric space $(\mathcal{X}, d)$ are reciprocally continuous if and only if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$ whenever $(x_n) \subset \mathcal{X}$ is such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in \mathcal{X}$. Clearly, any continuous pair is reciprocally continuous but, the converse is not true in general.

1.3 Definition *Two self-maps f and g of a metric space $(\mathcal{X}, d)$ are said to be* **subsequentially continuous** *if and only if there exists a sequence (x_n) in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in $\mathcal{X}$ and satisfy $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$.*

If f and g are both continuous or reciprocally continuous then they are obviously subsequentially continuous. The next example shows that there exist subsequentially continuous pairs of maps which are neither continuous nor reciprocally continuous.

1.4 Example Let $\mathcal{X}$ be $[0,\infty[$ endowed with the usual metric d and define f and $g : \mathcal{X} \to \mathcal{X}$ by

$$fx = \begin{cases} 1+x & \text{if} \quad 0 \le x \le 1 \\ 2x-1 & \text{if} \quad 1 < x < \infty, \end{cases} \quad gx = \begin{cases} 1-x & \text{if} \quad 0 \le x < 1 \\ 3x-2 & \text{if} \quad 1 \le x < \infty. \end{cases}$$

Obviously, f and g are discontinuous at $x = 1$.
Let us consider the sequence $x_n = \dfrac{1}{n}$ for $n = 1, 2, \dots$. We have

$$\begin{aligned} fx_n &= 1 + x_n \to 1 = t \quad && \text{when n} \to \infty, \\ gx_n &= 1 - x_n \to 1 && \text{when n} \to \infty, \end{aligned}$$

and

$$\begin{aligned} fgx_n &= f(1-x_n) = 2 - x_n \to 2 = f(1), \\ gfx_n &= g(1+x_n) = 1 + 3x_n \to 1 = g(1), \end{aligned}$$

therefore f and g are subsequentially continuous.
Now, let $(x_n) = (1 + \dfrac{1}{n})$ for $n = 1, 2, \dots$. We have

$$\begin{aligned} fx_n &= 2x_n - 1 \to 1 = t, \\ gx_n &= 3x_n - 2 \to 1 = t, \end{aligned}$$

and

$$fgx_n = f(3x_n - 2) = 6x_n - 5 \to 1 \ne 2 = f(1),$$

so f and g are not reciprocally continuous. ■

Now, we show the interest of these two definitions by giving three main results.

2 A general common fixed point theorem

We begin by a general common fixed point theorem which improves a result of [7].

2.1 Theorem *Let f, g, h and k be four self-maps of a metric space $(\mathcal{X}, d)$. If the pairs (f,h) and (g,k) are subcompatible and subsequentially continuous, then*
(a) f and h have a coincidence point;
(b) g and k have a coincidence point.
Further, let $\varphi : \mathbb{R}_+^6 \to \mathbb{R}$ be an upper semi-continuous function satisfying the following condition
$(\varphi_1) : \varphi(u,u,0,0,u,u) > 0 \ \forall u > 0.$
We suppose that (f,h) and (g,k) satisfy, for all x and y in $\mathcal{X}$,
$(\varphi_2) : \varphi(d(fx,gy), d(hx,ky), d(fx,hx), d(gy,ky), d(hx,gy), d(ky,fx)) \le 0.$
Then, f, g, h and k have a unique common fixed point.

Proof
Since the pairs (f,h) and (g,k) are subcompatible and subsequentially continuous, then, there exist two sequences (x_n) and (y_n) in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = t$ for some $t \in \mathcal{X}$ and which satisfy

$$\lim_{n\to\infty} d(fhx_n, hfx_n) = d(ft, ht) = 0;$$

$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} ky_n = t'$ for some $t' \in \mathcal{X}$ and which satisfy

$$\lim_{n\to\infty} d(gky_n, kgy_n) = d(gt', kt') = 0.$$

Therefore $ft = ht$ and $Bt' = Tt'$; that is, t is a coincidence point of f and h and t' is a coincidence point of g and k.
Now, we prove that $t = t'$. Indeed, by inequality (φ_2), we have

$$\begin{aligned} & \varphi(d(fx_n, gy_n), d(hx_n, ky_n), d(fx_n, hx_n), \\ & d(gy_n, ky_n), d(hx_n, gy_n), d(ky_n, fx_n)) \\ \leq \; & 0. \end{aligned}$$

Since φ is upper semi-continuous, taking the limit as $n \to \infty$ yields

$$\varphi(d(t,t'), d(t,t'), 0, 0, d(t,t'), d(t',t)) \leq 0$$

which contradicts (φ_1) if $t \neq t'$. Hence $t = t'$.

Also, we claim that $ft = t$. If $ft \neq t$, using (φ_2), we get

$$\begin{aligned} & \varphi(d(ft, gy_n), d(ht, ky_n), d(ft, ht), \\ & d(gy_n, ky_n), d(ht, gy_n), d(ky_n, ft)) \\ \leq \; & 0. \end{aligned}$$

Since φ is upper semi-continuous, at infinity, we get

$$\varphi(d(ft,t), d(ft,t), 0, 0, d(ft,t), d(t,ft)) \leq 0$$

contradicts (φ_1). Hence $t = ft = ht$.

Again, suppose that $gt \neq t$, using inequality (φ_2), we get

$$\begin{aligned} & \varphi(d(ft,gt), d(ht,kt), d(ft,ht), d(gt,kt), d(ht,gt), d(kt,ft)) \\ = \; & \varphi(d(t,gt), d(t,gt), 0, 0, d(t,gt), d(gt,t)) \leq 0 \end{aligned}$$

contradicts (φ_1). Thus $t = gt = kt$. Therefore $t = ft = gt = ht = kt$; i.e., $t = t'$ is a common fixed point of f, g, h and k.
Finally, suppose that there exists another common fixed point z of f, g, h and k such that $z \neq t$. Then, by inequality (φ_2), we have

$$\begin{aligned} & \varphi(d(ft,gz), d(ht,kz), d(ft,ht), d(gz,kz), d(ht,gz), d(kz,ft)) \\ = \; & \varphi(d(t,z), d(t,z), 0, 0, d(t,z), d(z,t)) \leq 0 \end{aligned}$$

which contradicts (φ_1). Hence $z = t$. ■

If we let in Theorem 2.1, $f = g$ and $h = k$, we get the next corollary

2.2 Corollary *Let f and h be self-maps of a metric space $(\mathcal{X}, d)$ such that the pair (f, h) is subcompatible and subsequentially continuous, then, f and h have a coincidence point.*
Further let $\varphi : \mathbb{R}_+^6 \to \mathbb{R}$ be an upper semi-continuous function satisfying condition (φ_1) and

$$\varphi(d(fx, fy), d(hx, hy), d(fx, hx), d(fy, hy), d(hx, fy), d(hy, fx)) \leq 0$$

for every x and every y in $\mathcal{X}$, then there exists a unique point $t \in \mathcal{X}$ such that $ft = ht = t$.

If we put $h = k$, we get the following result

2.3 Corollary *Let f, g and h be three self-maps of a metric space $(\mathcal{X}, d)$. Suppose that the pairs (f, h) and (g, h) are subcompatible and subsequentially continuous, then,*
(a) f and h have a coincidence point;
(b) g and h have a coincidence point.
Let $\varphi : \mathbb{R}_+^6 \to \mathbb{R}$ be an upper semi-continuous function satisfying condition (φ_1) and

$$\varphi(d(fx, gy), d(hx, hy), d(fx, hx), d(gy, hy), d(hx, gy), d(hy, fx)) \leq 0$$

for all x, y in $\mathcal{X}$, then f, g and h have a unique common fixed point $t \in \mathcal{X}$.

Now, with different choices of the real upper semi-continuous function φ, we obtain the following corollary which contains several already published results.

2.4 Corollary *If in the hypotheses of Theorem 2.1, we have instead of (φ_2) one of the following inequalities, for all x and y in $\mathcal{X}$, then the four maps have a unique common fixed point*

$$(a) \qquad \begin{aligned} d(fx, gy) \leq \; & \alpha \max\{d(hx, ky), d(hx, fx), d(gy, ky), \\ & \tfrac{1}{2}(d(hx, gy) + d(ky, fx))\} \end{aligned}$$

where $\alpha \in]0, 1[$

$$\begin{aligned} (b)\ d(fx, gy)&(1 + \alpha d(hx, ky)) \\ &\leq \alpha \max\{d(hx, fx)d(gy, ky), d(hx, gy)d(ky, fx)\} \\ &+\beta \max\{d(hx, ky), d(hx, fx), d(gy, ky), \tfrac{1}{2}(d(hx, gy) + d(ky, fx))\} \end{aligned}$$

where $\alpha \geq 0$ and $0 < \beta < 1$,

$$(c) \qquad d^3(fx, gy) \leq \frac{d^2(hx, fx)d^2(gy, ky) + d^2(hx, gy)d^2(ky, fx)}{1 + d(hx, ky) + d(hx, fx) + d(gy, ky)}$$

$$(d) \qquad d(fx,gy) \leq F[\max\{d(hx,ky),d(hx,fx), d(gy,ky),\tfrac{1}{2}(d(hx,gy)+d(ky,fx))\}]$$

where $F:\mathbb{R}_+\to\mathbb{R}_+$ *is an upper semi-continuous function such that, for every* $t>0$, $0<F(t)<t$.

Proof
For proof of (a), (b), (c) and (d), we use Theorem 2.1 with the next functions φ which satisfy, for every case, hypothesis (φ_1).
For (a):

$$\begin{aligned} & \varphi(d(fx,gy),d(hx,ky),d(fx,hx),d(gy,ky),d(hx,gy),d(ky,fx)) \\ = \ & d(fx,gy)-\alpha\max\{d(hx,ky),d(fx,hx),d(gy,ky), \\ & \tfrac{1}{2}(d(hx,gy)+d(ky,fx))\} \end{aligned}$$

this function φ is used by many authors, for example Example 1 of Popa [13].
For (b):

$$\begin{aligned} & \varphi(d(fx,gy),d(hx,ky),d(fx,hx),d(gy,ky),d(hx,gy),d(ky,fx)) \\ = \ & (1+\alpha d(hx,ky))d(fx,gy)-\alpha\max\{d(fx,hx)d(gy,ky), \\ & d(hx,gy)d(ky,fx)\}-\beta\max\{d(hx,ky),d(fx,hx),d(gy,ky), \\ & \tfrac{1}{2}(d(hx,gy)+d(ky,fx))\} \end{aligned}$$

for $\beta=1$, we have Example 3 of Popa [14].
For (c):

$$\begin{aligned} & \varphi(d(fx,gy),d(hx,ky),d(fx,hx),d(gy,ky),d(hx,gy),d(ky,fx)) \\ = \ & d^3(fx,gy)-\frac{d^2(fx,hx)d^2(gy,ky)+d^2(hx,gy)d^2(ky,fx)}{1+d(hx,ky)+d(fx,hx)+d(gy,ky)} \end{aligned}$$

this function φ is the one of Example 5 of [13] with $c=1$.
For (d):

$$\begin{aligned} & \varphi(d(fx,gy),d(hx,ky),d(fx,hx),d(gy,ky),d(hx,gy),d(ky,fx)) \\ = \ & d(fx,gy)-F[\max\{d(hx,ky),d(fx,hx),d(gy,ky), \\ & \tfrac{1}{2}(d(hx,gy)+d(ky,fx))\}]. \end{aligned}$$

■

Now, using the recurrence on n, we get the following theorem

2.5 Theorem *Let* h, k *and* $\{f_n\}_{n\in\mathbb{N}^*}$ *be maps from a metric space* $(\mathcal{X},d)$ *into itself such that the pairs* (f_n,h) *and* (f_{n+1},k) *are subcompatible and subsequentially continuous, then*
(a) (f_n,h) *have a coincidence point;*
(b) (f_{n+1},k) *have a coincidence point.*
Suppose that maps f_n, f_{n+1}, h *and* k *satisfy the inequality*

$$\begin{aligned} & \varphi(d(f_nx,f_{n+1}y),d(hx,ky),d(f_nx,hx), \\ & \qquad d(f_{n+1}y,ky),d(hx,f_{n+1}y),d(ky,f_nx)) \leq 0 \end{aligned}$$

for all x and y in $\mathcal{X}$, for every $n \in \mathbb{N}^*$, where φ is as in Theorem 2.1, then, h, k and $\{f_n\}_{n\in\mathbb{N}^*}$ have a unique common fixed point.

Proof
By letting $n = 1$, we get the assumptions of Theorem 2.1 for h, k, f_1 and f_2 with the unique common fixed point t. Now, t is a common fixed point of h, k, f_1 and of h, k, f_2. Otherwise, if t' is another common fixed point of h, k and f_1, then by inequality (ii), we have

$$\begin{gathered}\varphi(d(f_1t', f_2t), d(ht', kt), d(ht', f_1t'),\\ d(kt, f_2t), d(ht', f_2t), d(kt, f_1t'))\\ = \varphi(d(t', t), d(t', t), 0, 0, d(t', t), d(t, t')) \leq 0\end{gathered}$$

contradicts (φ_1), then $t' = t$.
By the same manner, we prove that t is the unique common fixed point of h, k and f_2.
Now, letting $n = 2$, we obtain hypotheses of Theorem 2.1 for h, k, f_2 and f_3 and then, they have a unique common fixed point t'. Analogously, t' is the unique common fixed point of h, k, f_2 and of h, k, f_3. Thus $t' = t$. Continuing by this method, we clearly see that t is the required element. ∎

2.6 Remark We can also have common fixed point by using only four distances instead of six. The next theorem shows this fact.

2.7 Theorem *Let f, g, h and k be self-maps of a metric space $(\mathcal{X}, d)$. If the pairs (f, h) and (g, k) are subcompatible and subsequentially continuous, then,*
(a) f and h have a coincidence point;
(b) g and k have a coincidence point.
Let $\varphi' : \mathbb{R}_+^4 \to \mathbb{R}$ be an upper semi-continuous function such that
$(\varphi_1') : \varphi'(u, u, u, u) > 0 \ \forall u > 0$.
Suppose that (f, h) and (g, k) satisfy the following inequality (φ_2'), for all x and y in $\mathcal{X}$,
$(\varphi_2') : \varphi'(d(fx, gy), d(hx, ky), d(hx, gy), d(ky, fx)) \leq 0$.
Then, f, g, h and k have a unique common fixed point.

Proof
First, proof of (a) and (b) is similar to proof of first part of Theorem 2.1.
Now, suppose that $d(t, t') > 0$, then, using inequality (φ_2'), we get

$$\varphi'(d(fx_n, gy_n), d(hx_n, ky_n), d(hx_n, gy_n), d(ky_n, fx_n)) \leq 0.$$

Since φ' is upper semi-continuous, we obtain at infinity

$$\varphi'(d(t, t'), d(t, t'), d(t, t'), d(t', t)) \leq 0$$

which contradicts (φ_1'), therefore $t' = t$.
If $d(ft, t) > 0$, by inequality (φ_2'), we have

$$\varphi'(d(ft, gy_n), d(ht, ky_n), d(ht, gy_n), d(ky_n, ft)) \leq 0.$$

Since φ' is upper semi-continuous, when n tends to infinity, we get

$$\varphi'(d(ft,t),d(ft,t),d(ft,t),d(t,ft)) \leq 0$$

which contradicts (φ'_1), hence $t = ft = ht$.
Similarly, we have $t = gt = kt$.
The uniqueness of the common fixed point t follows easily from inequality (φ'_2) and condition (φ'_1).

■

3 A type Greguš common fixed point theorem

In 1998, Pathak et al. [12] introduced an extension of compatibility of type (A) by giving the notion of compatibility of type (C) and they proved a common fixed point theorem of Greguš type for four compatible maps of type (C) in a Banach space. Further, Djoudi and Nisse [3] generalized the result of [12] by weakening compatibility of type (C) to weak compatibility without continuity. In 2006, Jungck and Rhoades [7] extended the result of Djoudi and Nisse by using an idea called occasionally weak compatibility of Al-Thagafi and Shahzad [2] which will be published in 2008.

In this part, we establish a common fixed point theorem for four subcompatible maps of Greguš type in a metric space which extends the results of [3], [7] and [12].

Let $\mathcal{F}$ be the family of maps F from $\mathbb{R}_+$ into itself such that F is upper semi-continuous and $F(t) < t$ for any $t > 0$.

3.1 Theorem *Let f, g, h and k be maps from a metric space $(\mathcal{X}, d)$ into itself. If the pairs (f,h) and (g,k) are subcompatible and subsequentially continuous, then,*
(a) (f,h) has a coincidence point;
(b) (g,k) has a coincidence point.
Moreover, suppose that the four maps satisfy the following inequality

$$(2)\ d^p(fx,gy) \leq F(ad^p(hx,ky) + (1-a)\max\{\alpha d^p(fx,hx), \beta d^p(gy,ky), d^{\frac{p}{2}}(fx,hx)d^{\frac{p}{2}}(fx,ky), d^{\frac{p}{2}}(fx,ky)d^{\frac{p}{2}}(hx,gy), \frac{1}{2}(d^p(fx,hx)+d^p(gy,ky))\})$$

for all x and y in $\mathcal{X}$, where $0 < a < 1$, $\{\alpha,\beta\} \subset]0,1]$, $p \in \mathbb{N}^*$ and $F \in \mathcal{F}$. Then f, g, h and k have a unique common fixed point.

Proof
As in proof of Theorem 2.1, we have $ft = ht$ and $gt' = kt'$. This ends the proofs of parts (a) and (b).

Furthermore, we prove that $t = t'$. Suppose that $d(t, t') > 0$, indeed by inequality (2) we have

$$\begin{aligned} d^p(fx_n, gy_n) \quad \leq \quad & F(ad^p(hx_n, ky_n) + (1-a)\max\{\alpha d^p(fx_n, hx_n), \\ & \beta d^p(gy_n, ky_n), d^{\frac{p}{2}}(fx_n, hx_n) d^{\frac{p}{2}}(fx_n, ky_n), \\ & d^{\frac{p}{2}}(fx_n, ky_n) d^{\frac{p}{2}}(hx_n, gy_n), \\ & \tfrac{1}{2}(d^p(fx_n, hx_n) + d^p(gy_n, ky_n))\}). \end{aligned}$$

By properties of F, we get at infinity

$$\begin{aligned} d^p(t, t') \quad & \leq \quad F(ad^p(t, t') + (1-a)d^p(t, t')) \\ & = \quad F(d^p(t, t')) < d^p(t, t') \end{aligned}$$

this contradiction implies that $t = t'$.
Now, if $ft \neq t$, the use of condition (2) gives

$$\begin{aligned} d^p(ft, gy_n) \quad \leq \quad & F(ad^p(ht, ky_n) + (1-a)\max\{\alpha d^p(ft, ht), \\ & \beta d^p(gy_n, ky_n), d^{\frac{p}{2}}(ft, ht) d^{\frac{p}{2}}(ft, ky_n), \\ & d^{\frac{p}{2}}(ft, ky_n) d^{\frac{p}{2}}(ht, gy_n), \\ & \tfrac{1}{2}(d^p(ft, ht) + d^p(gy_n, ky_n))\}). \end{aligned}$$

By properties of F, we obtain at infinity

$$\begin{aligned} d^p(ft, t) \quad & \leq \quad F(ad^p(ft, t) + (1-a)d^p(ft, t)) \\ & = \quad F(d^p(ft, t)) < d^p(ft, t) \end{aligned}$$

this contradiction implies that $t = ft = ht$.
Similarly, we have $gt = kt = t$. Therefore $t = t'$ is a common fixed point of both f, g, h and k.
Suppose that f, g, h and k have another common fixed point $z \neq t$. Then, by (2) we get

$$\begin{aligned} d^p(ft, gz) \quad \leq \quad & F(ad^p(ht, kz) + (1-a)\max\{\alpha d^p(ft, ht), \beta d^p(gz, kz), \\ & d^{\frac{p}{2}}(ft, ht) d^{\frac{p}{2}}(ft, kz), d^{\frac{p}{2}}(ft, kz) d^{\frac{p}{2}}(ht, gz), \\ & \tfrac{1}{2}(d^p(ft, ht) + d^p(gz, kz))\}); \end{aligned}$$

that is,

$$\begin{aligned} d^p(t, z) \quad & \leq \quad F(ad^p(t, z) + (1-a)\max\{0, d^p(t, z)\}) \\ & = \quad F(d^p(t, z)) < d^p(t, z) \end{aligned}$$

this contradiction implies that $z = t$. ∎

3.2 Corollary *Let f and h be two self-maps of a metric space $(\mathcal{X}, d)$. If the pair (f, h) is subcompatible and subsequentially continuous, then f and h have a coincidence point.*
Suppose that (f, h) satisfies the following inequality

$$\begin{aligned} d^p(fx, fy) \quad \leq \quad & F(ad^p(hx, hy) + (1-a)\max\{\alpha d^p(fx, hx), \beta d^p(fy, hy), \\ & d^{\frac{p}{2}}(fx, hx) d^{\frac{p}{2}}(fx, hy), d^{\frac{p}{2}}(fx, hy) d^{\frac{p}{2}}(hx, fy), \\ & \tfrac{1}{2}(d^p(fx, hx) + d^p(fy, hy))\}) \end{aligned}$$

for all x, y in $\mathcal{X}$, where $0 < a < 1$, $\{\alpha, \beta\} \subset]0,1]$, $p \in \mathbb{N}^*$ and $F \in \mathcal{F}$, then f and h have a unique common fixed point.

3.3 Corollary *Let f, g and h be three self-maps of a metric space $(\mathcal{X}, d)$. Suppose that the pairs (f,h) and (g,h) are subcompatible and subsequentially continuous, then,*
(a) f and h have a coincidence point;
(b) g and h have a coincidence point.
Further, if the three maps satisfy the next inequality

$$\begin{aligned} d^p(fx, gy) \leq\ & F(ad^p(hx,hy) + (1-a)\max\{\alpha d^p(fx,hx), \beta d^p(gy,hy), \\ & d^{\frac{p}{2}}(fx,hx)d^{\frac{p}{2}}(fx,hy), d^{\frac{p}{2}}(fx,hy)d^{\frac{p}{2}}(hx,gy), \\ & \tfrac{1}{2}(d^p(fx,hx) + d^p(gy,hy))\}) \end{aligned}$$

for all x and y in $\mathcal{X}$, where $0 < a < 1$, $\{\alpha, \beta\} \subset]0,1]$, $p \in \mathbb{N}^*$ and $F \in \mathcal{F}$, then f, g and h have a unique common fixed point.

Again, using the recurrence on n, we get the next theorem

3.4 Theorem *Let h, k and $\{f_n\}_{n\in\mathbb{N}^*}$ be self-maps of a metric space $(\mathcal{X}, d)$. Suppose that (f_n, h) and (f_{n+1}, k) are subcompatible and subsequentially continuous, then,*
(a) f_n and h have a coincidence point;
(b) f_{n+1} and k have a coincidence point.
Furthermore, if the maps satisfying the inequality

$$\begin{aligned} d^p(f_nx, f_{n+1}y) \leq\ & F\big(ad^p(hx,ky) + \\ & (1-a)\max\{\alpha d^p(f_nx,hx), \beta d^p(f_{n+1}y,ky), \\ & d^{\frac{p}{2}}(f_nx,hx)d^{\frac{p}{2}}(f_nx,ky), d^{\frac{p}{2}}(f_nx,ky)d^{\frac{p}{2}}(hx,f_{n+1}y), \\ & \tfrac{1}{2}(d^p(f_nx,hx) + d^p(f_{n+1}y,ky))\} \end{aligned}$$

for all x and y in $\mathcal{X}$, where $0 < a < 1$, $\{\alpha, \beta\} \subset]0,1]$, $p \in \mathbb{N}^*$ and $F \in \mathcal{F}$, then h, k and $\{f_n\}_{n\in\mathbb{N}^*}$ have a unique common fixed point.

4 A near-contractive common fixed point theorem

We end our work by establishing the next result which especially improves the main result of [8].

4.1 Theorem *Let $(\mathcal{X}, d)$ be a metric space, f, g, h and k be maps from $\mathcal{X}$ into itself. If the pairs (f,h) and (g,k) are subcompatible and subsequentially continuous, then,*
(a) f and h have a coincidence point;
(b) g and k have a coincidence point.

Let Φ be an upper semi-continuous function of $[0,\infty[$ into itself such that $\Phi(t) = 0$ iff $t = 0$ and satisfying inequality

$$\begin{aligned}(3)\ \Phi(d(fx,gy)) &\leq a(d(hx,ky))\Phi(d(hx,ky)) \\ &\quad +b(d(hx,ky))\min\{\Phi(d(hx,gy)),\Phi(d(ky,fx))\}\end{aligned}$$

for all x and y in $\mathcal{X}$, where a, $b : [0,\infty[\to [0,1[$ are upper semi-continuous and satisfying the condition

$$a(t) + b(t) < 1\ \forall t > 0.$$

Then f, g, h and k have a unique common fixed point.

Proof

First, proof of parts (a) and (b) is similar to proof of Theorem 2.1.
Now, suppose that $d(t,t') > 0$, using inequality (3), we get

$$\begin{aligned}&\Phi(d(fx_n,gy_n)) \\ \leq\ &a(d(hx_n,ky_n))\Phi(d(hx_n,ky_n)) \\ &+b(d(hx_n,ky_n))\min\{\Phi(d(hx_n,gy_n)),\Phi(d(ky_n,fx_n))\}.\end{aligned}$$

By properties of Φ, a and b, we get at infinity

$$\begin{aligned}\Phi(d(t,t')) &\leq [a(d(t,t')) + b(d(t,t'))]\Phi(d(t,t')) \\ &< \Phi(d(t,t'))\end{aligned}$$

which is a contradiction. Hence $\Phi(d(t,t')) = 0$ which implies that $d(t,t') = 0$, thus $t = t'$.
Next, if $ft \neq t$, the use of condition (3) gives

$$\begin{aligned}&\Phi(d(ft,gy_n)) \\ \leq\ &a(d(ht,ky_n))\Phi(d(ht,ky_n)) \\ &+b(d(ht,ky_n))\min\{\Phi(d(ht,gy_n)),\Phi(d(ky_n,ft))\}.\end{aligned}$$

By properties of Φ, a and b, we get at infinity

$$\begin{aligned}\Phi(d(ft,t)) &\leq [a(d(ft,t)) + b(d(ft,t))]\Phi(d(ft,t)) \\ &< \Phi(d(ft,t))\end{aligned}$$

this contradiction implies that $\Phi(d(ft,t)) = 0$ and hence $t = ft = ht$.
Similarly, we have $gt = kt = t$.
Now, assume that there exists another common fixed point z of f, g, h and k such that $z \neq t$. By inequality (3) and properties of functions Φ, a and b, we obtain

$$\begin{aligned}\Phi(d(t,z)) &= \Phi(d(ft,gz)) \leq a(d(ht,kz))\Phi(d(ht,kz)) \\ &\quad +b(d(ht,kz))\min\{\Phi(d(ht,gz)),\Phi(d(kz,ft))\} \\ &= [a(d(t,z)) + b(d(t,z))]\Phi(d(t,z)) \\ &< \Phi(d(t,z))\end{aligned}$$

this contradiction implies that $\Phi(d(t,z)) = 0 \Leftrightarrow d(t,z) = 0$, hence $z = t$. ■

4.2 Remark A slight improvement is achieved by replacing inequality (3) in Theorem 4.1 by the following one

$$\begin{aligned}\Phi(d(fx,gy)) \quad &\leq \quad a(d(hx,ky))\Phi(d(hx,ky)) \\ &\quad +b(d(hx,ky))\left[\frac{\Phi^{\frac{1}{p}}(d(hx,gy))+\Phi^{\frac{1}{p}}(d(ky,fx))}{2}\right]^p,\end{aligned}$$

where p is an integer such that $p \geq 1$.

As particular cases, we immediately obtain the two following corollaries

4.3 Corollary *Let f and h be self-maps of a metric space $(\mathcal{X}, d)$. Assume that the pair (f,h) is subcompatible and subsequentially continuous, then, f and h have a coincidence point.*
Further, suppose that the pair (f,h) satisfies the inequality

$$\begin{aligned}\Phi(d(fx,fy)) \quad &\leq \quad a(d(hx,hy))\Phi(d(hx,hy)) \\ &\quad +b(d(hx,hy))\min\{\Phi(d(hx,fy)),\Phi(d(hy,fx))\}\end{aligned}$$

for all x and y in $\mathcal{X}$, where Φ, a and b are as in Theorem 4.1. Then, f and h have a unique common fixed point.

4.4 Corollary *Let f, g, $h : \mathcal{X} \to \mathcal{X}$ be maps. If the pairs (f,h) and (g,h) are subcompatible and subsequentially continuous, then,*
(a) f and h have a coincidence point;
(b) g and h have a coincidence point.
Moreover, suppose that maps f, g and h satisfy the following inequality

$$\begin{aligned}\Phi(d(fx,gy)) \quad &\leq \quad a(d(hx,hy))\Phi(d(hx,hy)) \\ &\quad +b(d(hx,hy))\min\{\Phi(d(hx,gy)),\Phi(d(hy,fx))\}\end{aligned}$$

for all x and y in $\mathcal{X}$, where Φ, a and b are as in Theorem 4.1, then, f, g and h have a unique common fixed point.

We end our work by giving the following result which concern a common fixed point of a sequence of maps. Its proof is easily obtained from Theorem 4.1 by recurrence.

4.5 Theorem *Let $(\mathcal{X}, d)$ be a metric space, h, k, $\{f_n\}_{n\in\mathbb{N}^*}$ be maps from $\mathcal{X}$ into itself. If the pairs (f_n, h) and (f_{n+1}, k) are subcompatible and subsequentially continuous, then,*
(a) f_n and h have a coincidence point;
(b) f_{n+1} and k have a coincidence point.
Let Φ be an upper semi-continuous function of $[0,\infty[$ into itself such that $\Phi(t) = 0$ iff $t = 0$ and satisfying the following inequality

$$\begin{aligned}\Phi(d(f_nx,f_{n+1}y)) \quad &\leq \quad a(d(hx,ky))\Phi(d(hx,ky)) \\ &\quad +b(d(hx,ky))\min\{\Phi(d(hx,f_{n+1}y)),\Phi(d(ky,f_nx))\}\end{aligned}$$

for all x and y in $\mathcal{X}$, where a and $b : [0,\infty[\to [0,1[$ are upper semi-continuous and satisfying the condition

$$a(t) + b(t) < 1 \ \ \forall t > 0.$$

Then, h, k and $\{f_n\}_{n\in\mathbb{N}^*}$ have a unique common fixed point.

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